Supporting Information

I. DERIVING BERRY'S FORCE FROM THE QCLE

A. The $\frac{\partial A_{11}^W}{\partial t}$ Expression

To derive the $\frac{\partial A_{11}^W}{\partial t}$ expression (Eq. 5), we start from the definition of ζ (Eq. 4) in the paper

$$\zeta = \frac{i\hbar\Delta}{V_{11}-V_{22}}\frac{P^\gamma}{M^\gamma}d_{12}^\gamma + \frac{i\hbar}{2}d_{12}^\alpha\frac{\partial\Gamma}{\partial P^\alpha} \tag{S1}$$

$$\Delta \equiv A_{22}^W - A_{11}^W \tag{S2}$$

$$\Gamma \equiv A_{22}^W + A_{11}^W \tag{S3}$$

$$B_{12}^W \equiv A_{12}^W - \zeta \tag{S4}$$

Using Eq. S4, Eq. 1 in the paper becomes

$$\begin{aligned}\frac{\partial A_{11}^W}{\partial t} &= \frac{2P^\alpha}{M^\alpha}Re\left(\left(B_{12}^W+\zeta\right)d_{21}^\alpha\right) - \frac{P^\alpha}{M^\alpha}\frac{\partial A_{11}^W}{\partial R^\alpha} - F_{11}^\alpha\frac{\partial A_{11}^W}{\partial P^\alpha} - Re\left(\frac{\partial\left(B_{12}^W+\zeta\right)}{\partial P^\alpha}F_{21}^\alpha\right) \\ &= \frac{2P^\alpha}{M^\alpha}Re\left(B_{12}^W d_{21}^\alpha\right) - \frac{P^\alpha}{M^\alpha}\frac{\partial A_{11}^W}{\partial R^\alpha} - F_{11}^\alpha\frac{\partial A_{11}^W}{\partial P^\alpha} - Re\left(\frac{\partial B_{12}^W}{\partial P^\alpha}F_{21}^\alpha\right) \\ &\quad + \frac{2P^\alpha}{M^\alpha}Re\left(\zeta d_{21}^\alpha\right) - Re\left(\frac{\partial\zeta}{\partial P^\alpha}F_{21}^\alpha\right)\end{aligned} \tag{S5}$$

We focus on the last 2 terms above:

$$\begin{aligned}\frac{2P^\alpha}{M^\alpha}Re\left(\zeta d_{21}^\alpha\right) &= Re\left(\frac{i\hbar\Delta}{V_{11}-V_{22}}\frac{P^\gamma}{M^\gamma}d_{12}^\gamma\frac{P^\alpha}{M^\alpha}d_{21}^\alpha\right) + Re\left(\frac{i\hbar}{2}d_{12}^\gamma\frac{\partial\Gamma}{\partial P^\gamma}\frac{P^\alpha}{M^\alpha}d_{21}^\alpha\right) \\ &= 0 + Re\left(\frac{i\hbar}{2}d_{12}^\gamma\frac{P^\alpha}{M^\alpha}d_{21}^\alpha\right)\frac{\partial\Gamma}{\partial P^\gamma} \\ &= \hbar Im\left(d_{21}^\gamma\frac{P^\alpha}{M^\alpha}d_{12}^\alpha\right)\frac{\partial\Gamma}{\partial P^\gamma} \\ &= \hbar Im\left(d_{21}^\alpha\frac{P^\gamma}{M^\gamma}d_{12}^\gamma\right)\frac{\partial\Gamma}{\partial P^\alpha}\end{aligned} \tag{S6}$$

For the last term:

$$
\begin{aligned}
-Re\left(\frac{\partial\zeta}{\partial P^\alpha}F_{21}^\alpha\right) &= -\frac{1}{2}Re\left(-i\hbar\frac{P^\gamma}{M^\gamma}d_{12}^\gamma\frac{\partial\Delta}{\partial P^\alpha}d_{21}^\alpha\right) \\
&\quad -\frac{1}{2}Re\left(-i\hbar\Delta\frac{d_{12}^\alpha d_{21}^\alpha}{M^\alpha}\right) \\
&\quad -\frac{1}{2}Re\left(\frac{i\hbar}{2}d_{12}^\gamma\frac{\partial^2\Gamma}{\partial P^\gamma\partial P^\alpha}F_{21}^\alpha\right) \\
&= -\frac{1}{2}Re\left(i\hbar d_{12}^\alpha\frac{P^\gamma}{M^\gamma}d_{21}^\gamma\right)\frac{\partial\Delta}{\partial P^\alpha}+0+0 \\
&= \hbar Im\left(d_{12}^\alpha\frac{P^\gamma}{M^\gamma}d_{21}^\gamma\right)\frac{\partial\Delta}{\partial P^\alpha}
\end{aligned}
\tag{S7}
$$

Combining the two terms, we find:

$$
\begin{aligned}
\frac{2P^\alpha}{M^\alpha}Re\left(\zeta d_{21}^\alpha\right)-Re\left(\frac{\partial\zeta}{\partial P^\alpha}F_{21}^\alpha\right) &= \hbar Im\left(d_{21}^\alpha\frac{P^\gamma}{M^\gamma}d_{12}^\gamma\right)\frac{\partial\Gamma}{\partial P^\alpha}+\hbar Im\left(d_{12}^\alpha\frac{P^\gamma}{M^\gamma}d_{21}^\gamma\right)\frac{\partial\Delta}{\partial P^\alpha} \\
&= \hbar Im\left(d_{12}^\alpha\frac{P^\gamma}{M^\gamma}d_{21}^\gamma\right)\frac{\partial(\Delta-\Gamma)}{\partial P^\alpha} \\
&= -2\hbar Im\left(d_{12}^\alpha\frac{P^\gamma}{M^\gamma}d_{21}^\gamma\right)\frac{\partial A_{11}^W}{\partial P^\alpha} \\
&= 2\hbar Im\left(d_{21}^\gamma\frac{P^\alpha}{M^\alpha}d_{12}^\alpha\right)\frac{\partial A_{11}^W}{\partial P^\gamma}
\end{aligned}
\tag{S8}
$$

Therefore, we find:

$$
\begin{aligned}
\frac{\partial A_{11}^W}{\partial t} &= \frac{2P^\alpha}{M^\alpha}Re\left(B_{12}^W d_{21}^\alpha\right)-\frac{P^\alpha}{M^\alpha}\frac{\partial A_{11}^W}{\partial R^\alpha}-F_{11}^\alpha\frac{\partial A_{11}^W}{\partial P^\alpha}-Re\left(\frac{\partial B_{12}^W}{\partial P^\alpha}F_{21}^\alpha\right) \\
&\quad +2\hbar Im\left(d_{21}^\gamma\frac{P^\alpha}{M^\alpha}d_{12}^\alpha\right)\frac{\partial A_{11}^W}{\partial P^\gamma}
\end{aligned}
\tag{S9}
$$

This is the A_{11}^W result.

B. The $\frac{\partial B_{12}^W}{\partial t}$ Expression

Now let's consider $B_{12}^W \equiv A_{12}^W - \zeta$:

$$\begin{aligned}
\frac{\partial B_{12}^W}{\partial t} &= \frac{\partial A_{12}^W}{\partial t} - \frac{\partial \zeta}{\partial t} \\
&= \frac{-i}{\hbar}\left(V_{11}-V_{22}\right)A_{12}^W - \frac{P^\alpha}{M^\alpha}d_{12}^\alpha\Delta - \frac{P^\alpha}{M^\alpha}\frac{\partial A_{12}^W}{\partial R^\alpha} \\
&\quad -\frac{1}{2}\left(F_{11}^\alpha + F_{22}^\alpha\right)\frac{\partial A_{12}^W}{\partial P^\alpha} - \frac{1}{2}F_{12}^\alpha\frac{\partial \Gamma}{\partial P^\alpha} - \frac{\partial \zeta}{\partial t} \\
&= -\frac{i}{\hbar}\left(V_{11}-V_{22}\right)B_{12}^W - \frac{P^\alpha}{M^\alpha}d_{12}^\alpha\Delta - \frac{P^\alpha}{M^\alpha}\frac{\partial B_{12}^W}{\partial R^\alpha} \\
&\quad -\frac{1}{2}\left(F_{11}^\alpha + F_{22}^\alpha\right)\frac{\partial B_{12}^W}{\partial P^\alpha} - \frac{1}{2}F_{12}^\alpha\frac{\partial \Gamma}{\partial P^\alpha} \\
&\quad -\frac{i}{\hbar}\left(V_{11}-V_{22}\right)\zeta - \frac{P^\alpha}{M^\alpha}\frac{\partial \zeta}{\partial R^\alpha} - \frac{1}{2}\left(F_{11}^\alpha + F_{22}^\alpha\right)\frac{\partial \zeta}{\partial P^\alpha} - \frac{\partial \zeta}{\partial t}
\end{aligned} \tag{S10}$$

From the definition of ζ we have

$$-\frac{i}{\hbar}(V_{11}-V_{22})\zeta - \frac{P^\alpha}{M^\alpha}d_{12}^\alpha\Delta - \frac{1}{2}F_{12}^\alpha\frac{\partial \Gamma}{\partial P^\alpha} = 0 \tag{S11}$$

and so three terms vanish in Eq. S10, leading to:

$$\begin{aligned}
\frac{\partial B_{12}^W}{\partial t} &= -\frac{i}{\hbar}\left(V_{11}-V_{22}\right)B_{12}^W - \frac{P^\alpha}{M^\alpha}\frac{\partial B_{12}^W}{\partial R^\alpha} - \frac{1}{2}\left(F_{11}^\alpha + F_{22}^\alpha\right)\frac{\partial B_{12}^W}{\partial P^\alpha} \\
&\quad -\frac{1}{2}\left(F_{11}^\alpha + F_{22}^\alpha\right)\frac{\partial \zeta}{\partial P^\alpha} - \frac{P^\alpha}{M^\alpha}\frac{\partial \zeta}{\partial R^\alpha} - \frac{\partial \zeta}{\partial t}
\end{aligned} \tag{S12}$$

We must now evaluate the last three terms in Eq. S12

The first term is

$$\begin{aligned}
-\frac{1}{2}\left(F_{11}^\alpha + F_{22}^\alpha\right)\frac{\partial \zeta}{\partial P^\alpha} &= -\frac{1}{2}\left(F_{11}^\alpha + F_{22}^\alpha\right)\left(\frac{i\hbar}{V_{11}-V_{22}}\frac{\partial \Delta}{\partial P^\alpha}\frac{P^\gamma}{M^\gamma}d_{12}^\gamma + \frac{i\hbar\Delta}{V_{11}-V_{22}}\frac{d_{12}^\alpha}{M^\alpha} + \frac{i\hbar}{2}d_{12}^\gamma\frac{\partial^2\Gamma}{\partial P^\gamma \partial P^\alpha}\right) \\
&= -\frac{i\hbar}{2}(F_{11}^\alpha + F_{22}^\alpha)\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\frac{\partial \Delta}{\partial P^\alpha} \\
&\quad -\frac{i\hbar}{2}(F_{11}^\alpha + F_{22}^\alpha)\frac{d_{12}^\alpha\Delta}{M^\alpha(V_{11}-V_{22})} \\
&\quad -\frac{i\hbar}{4}(F_{11}^\alpha + F_{22}^\alpha)d_{12}^\gamma\frac{\partial^2\Gamma}{\partial P^\gamma \partial P^\alpha}
\end{aligned} \tag{S13}$$

The second term is

$$
\begin{aligned}
-\frac{P^\alpha}{M^\alpha}\frac{\partial\zeta}{\partial R^\alpha} &= -\frac{P^\alpha}{M^\alpha}\left(\frac{i\hbar}{V_{11}-V_{22}}\frac{P^\gamma}{M^\gamma}d_{12}^\gamma\frac{\partial\Delta}{\partial R^\alpha}+\frac{i\hbar\Delta(F_{11}^\alpha-F_{22}^\alpha)}{(V_{11}-V_{22})^2}\frac{P^\gamma}{M^\gamma}d_{12}^\gamma\right. \\
&\quad \left.+\frac{i\hbar\Delta}{V_{11}-V_{22}}\frac{P^\gamma}{M^\gamma}\frac{\partial d_{12}^\gamma}{\partial R^\alpha}+\frac{i\hbar}{2}\frac{\partial d_{12}^\gamma}{\partial R^\alpha}\frac{\partial\Gamma}{\partial P^\gamma}+\frac{i\hbar}{2}d_{12}^\gamma\frac{\partial^2\Gamma}{\partial P^\gamma\partial R^\alpha}\right) \\
&= -i\hbar\frac{P^\alpha}{M^\alpha}\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\frac{\partial\Delta}{\partial R^\alpha} \\
&\quad -i\hbar\frac{P^\alpha}{M^\alpha}\frac{P^\gamma d_{12}^\gamma}{M^\gamma}\frac{\Delta}{(V_{11}-V_{22})^2}(F_{11}^\alpha-F_{22}^\alpha) \\
&\quad -i\hbar\frac{P^\alpha}{M^\alpha}\frac{\partial d_{12}^\gamma}{\partial R^\alpha}\frac{P^\gamma}{M^\gamma}\frac{\Delta}{V_{11}-V_{22}} \\
&\quad -\frac{i\hbar}{2}\frac{P^\alpha}{M^\alpha}\frac{\partial d_{12}^\gamma}{\partial R^\alpha}\frac{\partial\Gamma}{\partial P^\gamma} \\
&\quad -\frac{i\hbar}{2}\frac{P^\alpha}{M^\alpha}d_{12}^\gamma\frac{\partial^2\Gamma}{\partial P^\gamma\partial R^\alpha}
\end{aligned}
\tag{S14}
$$

The third term is

$$
\begin{aligned}
-\frac{\partial\zeta}{\partial t} &= -i\hbar\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\frac{\partial\Delta}{\partial t}-\frac{i\hbar}{2}d_{12}^\gamma\frac{\partial}{\partial t}\frac{\partial\Gamma}{\partial P^\gamma} \\
&= -i\hbar\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\frac{\partial\Delta}{\partial t}-\frac{i\hbar}{2}d_{12}^\gamma\frac{\partial}{\partial P^\gamma}\frac{\partial\Gamma}{\partial t}
\end{aligned}
\tag{S15}
$$

Now we must evaluate $\frac{\partial\Delta}{\partial t}$ and $\frac{\partial\Gamma}{\partial t}$. Applying

$$
\begin{aligned}
\frac{\partial A_{11}^W}{\partial t} &= \frac{2P^\alpha}{M^\alpha}Re\left(A_{12}^W d_{21}^\alpha\right)-\frac{P^\alpha}{M^\alpha}\frac{\partial A_{11}^W}{\partial R^\alpha}-F_{11}^\alpha\frac{\partial A_{11}^W}{\partial P^\alpha}-Re\left(\frac{\partial A_{12}^W}{\partial P^\alpha}F_{21}^\alpha\right) \\
\frac{\partial A_{22}^W}{\partial t} &= \frac{2P^\alpha}{M^\alpha}Re\left(A_{21}^W d_{12}^\alpha\right)-\frac{P^\alpha}{M^\alpha}\frac{\partial A_{22}^W}{\partial R^\alpha}-F_{22}^\alpha\frac{\partial A_{22}^W}{\partial P^\alpha}-Re\left(\frac{\partial A_{21}^W}{\partial P^\alpha}F_{12}^\alpha\right)
\end{aligned}
\tag{S16}
$$

we find:

$$
\begin{aligned}
\frac{\partial\Delta}{\partial t} &= -\frac{4P^\alpha}{M^\alpha}Re\left(A_{12}^W d_{21}^\alpha\right)-\frac{P^\alpha}{M^\alpha}\frac{\partial\Delta}{\partial R^\alpha}-F_{22}^\alpha\frac{\partial A_{22}^W}{\partial P^\alpha}+F_{11}^\alpha\frac{\partial A_{11}^W}{\partial P^\alpha} \\
\frac{\partial\Gamma}{\partial t} &= -\frac{P^\alpha}{M^\alpha}\frac{\partial\Gamma}{\partial R^\alpha}-\left(F_{11}^\alpha\frac{\partial A_{11}^W}{\partial P^\alpha}+F_{22}^\alpha\frac{\partial A_{22}^W}{\partial P^\alpha}\right)-2Re\left(\frac{\partial A_{12}^W}{\partial P^\alpha}F_{21}^\alpha\right)
\end{aligned}
\tag{S17}
$$

$$
\frac{\partial}{\partial P^\gamma}\frac{\partial\Gamma}{\partial t} = -\frac{\partial}{\partial P^\gamma}\left(\frac{P^\alpha}{M^\alpha}\frac{\partial\Gamma}{\partial R^\alpha}+\left(F_{22}^\alpha\frac{\partial A_{11}^W}{\partial P^\alpha}+F_{22}^\alpha\frac{\partial A_{22}^W}{\partial P^\alpha}\right)+2Re\left(\frac{\partial A_{12}^W}{\partial P^\alpha}F_{21}^\alpha\right)\right) \tag{S18}
$$

Thus

$$
\begin{aligned}
-\frac{\partial \zeta}{\partial t} &= -i\hbar \frac{P^\gamma d_{12}^\gamma}{M^\gamma (V_{11}-V_{22})} \frac{\partial \Delta}{\partial t} - \frac{i\hbar}{2} d_{12}^\gamma \frac{\partial}{\partial P^\gamma} \frac{\partial \Gamma}{\partial t} \\
&= i\hbar \frac{P^\gamma d_{12}^\gamma}{M^\gamma (V_{11}-V_{22})} \left(\frac{4P^\alpha}{M^\alpha} Re\left(A_{12}^W d_{21}^\alpha \right) \right) \\
&\quad + i\hbar \frac{P^\gamma d_{12}^\gamma}{M^\gamma (V_{11}-V_{22})} \frac{P^\alpha}{M^\alpha} \frac{\partial \Delta}{\partial R^\alpha} \\
&\quad + i\hbar \frac{P^\gamma d_{12}^\gamma}{M^\gamma (V_{11}-V_{22})} \left(F_{22}^\alpha \frac{\partial A_{22}^W}{\partial P^\alpha} - F_{11}^\alpha \frac{\partial A_{11}^W}{\partial P^\alpha} \right) \\
&\quad + \frac{i\hbar}{2} \frac{d_{12}^\gamma P^\alpha}{M^\alpha} \frac{\partial^2 \Gamma}{\partial R^\alpha \partial P^\gamma} \\
&\quad + \frac{i\hbar}{2} \frac{d_{12}^\gamma}{M^\gamma} \frac{\partial \Gamma}{\partial R^\gamma} \\
&\quad + \frac{i\hbar}{2} d_{12}^\gamma \left(F_{11}^\alpha \frac{\partial^2 A_{11}^W}{\partial P^\alpha \partial P^\gamma} + F_{22}^\alpha \frac{\partial^2 A_{22}^W}{\partial P^\alpha \partial P^\gamma} \right) \\
&\quad + i\hbar d_{12}^\gamma Re\left(\frac{\partial^2 A_{12}^W}{\partial P^\alpha \partial P^\gamma} F_{21}^\alpha \right)
\end{aligned}
\tag{S19}
$$

Putting all three terms together, we find:

$$
\begin{aligned}
\frac{\partial B_{12}^W}{\partial t} \;=\; & -\frac{i}{\hbar}\left(V_{11}-V_{22}\right)B_{12}^W - \frac{P^\alpha}{M^\alpha}\frac{\partial B_{12}^W}{\partial R^\alpha} - \frac{1}{2}\left(F_{11}^\alpha + F_{22}^\alpha\right)\frac{\partial B_{12}^W}{\partial P^\alpha} \\
& -\frac{i\hbar}{2}(F_{11}^\alpha + F_{22}^\alpha)\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\frac{\partial \Delta}{\partial P^\alpha} \\
& -\frac{i\hbar}{2}(F_{11}^\alpha + F_{22}^\alpha)\frac{d_{12}^\alpha \Delta}{M^\alpha(V_{11}-V_{22})} \\
& -\frac{i\hbar}{4}(F_{11}^\alpha + F_{22}^\alpha)d_{12}^\gamma\frac{\partial^2\Gamma}{\partial P^\gamma \partial P^\alpha} \\
& -i\hbar\frac{P^\alpha}{M^\alpha}\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\frac{\partial \Delta}{\partial R^\alpha} \\
& -i\hbar\frac{P^\alpha}{M^\alpha}\frac{P^\gamma d_{12}^\gamma}{M^\gamma}\frac{\Delta}{(V_{11}-V_{22})^2}(F_{11}^\alpha - F_{22}^\alpha) \\
& -i\hbar\frac{P^\alpha}{M^\alpha}\frac{\partial d_{12}^\gamma}{\partial R^\alpha}\frac{P^\gamma}{M^\gamma}\frac{\Delta}{V_{11}-V_{22}} \\
& -\frac{i\hbar}{2}\frac{P^\alpha}{M^\alpha}\frac{\partial d_{12}^\gamma}{\partial R^\alpha}\frac{\partial \Gamma}{\partial P^\gamma} \\
& -\frac{i\hbar}{2}\frac{P^\alpha}{M^\alpha}d_{12}^\gamma\frac{\partial^2\Gamma}{\partial P^\gamma \partial R^\alpha} \\
& +i\hbar\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\left(\frac{4P^\alpha}{M^\alpha}Re\left(A_{12}^W d_{21}^\alpha\right)\right) \\
& +i\hbar\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\frac{P^\alpha}{M^\alpha}\frac{\partial \Delta}{\partial R^\alpha} \\
& +i\hbar\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\left(F_{22}^\alpha\frac{\partial A_{22}^W}{\partial P^\alpha} - F_{11}^\alpha\frac{\partial A_{11}^W}{\partial P^\alpha}\right) \\
& +\frac{i\hbar}{2}\frac{d_{12}^\gamma P^\alpha}{M^\alpha}\frac{\partial^2\Gamma}{\partial R^\alpha \partial P^\gamma} \\
& +\frac{i\hbar}{2}\frac{d_{12}^\gamma}{M^\gamma}\frac{\partial \Gamma}{\partial R^\gamma} \\
& +\frac{i\hbar}{2}d_{12}^\gamma\left(F_{11}^\alpha\frac{\partial^2 A_{11}^W}{\partial P^\alpha \partial P^\gamma} + \frac{F_{22}^\alpha \partial^2 A_{22}^W}{\partial P^\alpha \partial P^\gamma}\right) \\
& +i\hbar d_{12}^\gamma Re\left(\frac{\partial^2 A_{12}^W}{\partial P^\alpha \partial P^\gamma}F_{21}^\alpha\right)
\end{aligned}
\tag{S20}
$$

Next, we eliminate and combine a few terms to find:

$$
\begin{aligned}
\frac{\partial B_{12}^W}{\partial t} \;=\; & -\frac{i}{\hbar}\left(V_{11}-V_{22}\right)B_{12}^W - \frac{P^\alpha}{M^\alpha}\frac{\partial B_{12}^W}{\partial R^\alpha} - \frac{1}{2}\left(F_{11}^\alpha + F_{22}^\alpha\right)\frac{\partial B_{12}^W}{\partial P^\alpha} \\
& -\frac{i\hbar}{2}(F_{11}^\alpha + F_{22}^\alpha)\frac{d_{12}^\alpha \Delta}{M^\alpha(V_{11}-V_{22})} \\
& -i\hbar\frac{P^\alpha}{M^\alpha}\frac{P^\gamma d_{12}^\gamma}{M^\gamma}\frac{\Delta}{(V_{11}-V_{22})^2}(F_{11}^\alpha - F_{22}^\alpha) \\
& -i\hbar\frac{P^\alpha}{M^\alpha}\frac{\partial d_{12}^\gamma}{\partial R^\alpha}\frac{P^\gamma}{M^\gamma}\frac{\Delta}{V_{11}-V_{22}} \\
& -\frac{i\hbar}{2}\frac{P^\alpha}{M^\alpha}\frac{\partial d_{12}^\gamma}{\partial R^\alpha}\frac{\partial \Gamma}{\partial P^\gamma} \\
& +i\hbar\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\left(\frac{4P^\alpha}{M^\alpha}Re\left(A_{12}^W d_{21}^\alpha\right)\right) \\
& +\frac{i\hbar}{2}\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\left((F_{22}^\alpha - F_{11}^\alpha)\frac{\partial \Delta}{\partial P^\alpha}\right) \\
& +\frac{i\hbar}{2}\frac{d_{12}^\gamma}{M^\gamma}\frac{\partial \Gamma}{\partial R^\gamma} \\
& +\frac{i\hbar}{4}d_{12}^\gamma\left((F_{22}^\alpha - F_{11}^\alpha)\frac{\partial^2 \Delta}{\partial P^\alpha \partial P^\gamma}\right) \\
& +i\hbar d_{12}^\gamma Re\left(\frac{\partial^2 A_{12}^W}{\partial P^\alpha \partial P^\gamma}F_{21}^\alpha\right)
\end{aligned}
\tag{S21}
$$

Lastly, we convert $A_{12}^W \to B_{12}^W + \zeta$

$$\begin{aligned}
\frac{\partial B_{12}^W}{\partial t} \;=\; & -\frac{i}{\hbar}\left(V_{11}-V_{22}\right)B_{12}^W - \frac{P^\alpha}{M^\alpha}\frac{\partial B_{12}^W}{\partial R^\alpha} - \frac{1}{2}\left(F_{11}^\alpha + F_{22}^\alpha\right)\frac{\partial B_{12}^W}{\partial P^\alpha} \\
& -\frac{i\hbar}{2}(F_{11}^\alpha + F_{22}^\alpha)\frac{d_{12}^\alpha \Delta}{M^\alpha(V_{11}-V_{22})} \\
& -i\hbar\frac{P^\alpha}{M^\alpha}\frac{P^\gamma d_{12}^\gamma}{M^\gamma}\frac{\Delta}{(V_{11}-V_{22})^2}(F_{11}^\alpha - F_{22}^\alpha) \\
& -i\hbar\frac{P^\alpha}{M^\alpha}\frac{\partial d_{12}^\gamma}{\partial R^\alpha}\frac{P^\gamma}{M^\gamma}\frac{\Delta}{V_{11}-V_{22}} \\
& -\frac{i\hbar}{2}\frac{P^\alpha}{M^\alpha}\frac{\partial d_{12}^\gamma}{\partial R^\alpha}\frac{\partial \Gamma}{\partial P^\gamma} \\
& +i\hbar\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\left(\frac{4P^\alpha}{M^\alpha}Re\left(B_{12}^W d_{21}^\alpha\right)\right) \\
& +\frac{i\hbar}{2}\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\left((F_{22}^\alpha - F_{11}^\alpha)\frac{\partial \Delta}{\partial P^\alpha}\right) \\
& +\frac{i\hbar}{2}\frac{d_{12}^\gamma}{M^\gamma}\frac{\partial \Gamma}{\partial R^\gamma} \\
& +\frac{i\hbar}{4}d_{12}^\gamma\left((F_{22}^\alpha - F_{11}^\alpha)\frac{\partial^2 \Delta}{\partial P^\alpha \partial P^\gamma}\right) \\
& +i\hbar d_{12}^\gamma Re\left(\frac{\partial^2 B_{12}^W}{\partial P^\alpha \partial P^\gamma}F_{21}^\alpha\right) \\
& +i\hbar\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\left(\frac{4P^\alpha}{M^\alpha}Re\left(\zeta d_{21}^\alpha\right)\right) \\
& +i\hbar d_{12}^\gamma Re\left(\frac{\partial^2 \zeta}{\partial P^\alpha \partial P^\gamma}F_{21}^\alpha\right)
\end{aligned} \tag{S22}$$

The last two terms above can be rewritten according to Eqs. S6 and S7

$$\begin{aligned}
& i\hbar\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\left(\frac{4P^\alpha}{M^\alpha}Re\left(\zeta d_{21}^\alpha\right)\right) + i\hbar d_{12}^\gamma Re\left(\frac{\partial^2 \zeta}{\partial P^\alpha \partial P^\gamma}F_{21}^\alpha\right) \\
= \; & i\hbar\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\left[2\hbar Im\left(d_{21}^\alpha\frac{P^\gamma}{M^\gamma}d_{12}^\gamma\right)\frac{\partial \Gamma}{\partial P^\alpha}\right] - i\hbar d_{12}^\gamma\frac{\partial}{\partial P^\gamma}\left[\hbar Im\left(d_{12}^\alpha\frac{P^\gamma}{M^\gamma}d_{21}^\gamma\right)\frac{\partial \Delta}{\partial P^\alpha}\right] \\
= \; & i\hbar\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\left[2\hbar Im\left(d_{21}^\alpha\frac{P^\gamma}{M^\gamma}d_{12}^\gamma\right)\frac{\partial \Gamma}{\partial P^\alpha}\right] \\
& -i\hbar d_{12}^\gamma\left[\hbar Im\left(\frac{d_{12}^\alpha d_{21}^\gamma}{M^\gamma}\right)\frac{\partial \Delta}{\partial P^\alpha}\right] \\
& -i\hbar d_{12}^\gamma\left[\hbar Im\left(d_{12}^\alpha\frac{P^\gamma}{M^\gamma}d_{21}^\gamma\right)\frac{\partial^2 \Delta}{\partial P^\alpha \partial P^\gamma}\right]
\end{aligned} \tag{S23}$$

In the end, we arrive at the following expression for $\frac{\partial B_{12}^W}{\partial t}$:

$$
\begin{aligned}
\frac{\partial B_{12}^W}{\partial t} \;=\; & -\frac{i}{\hbar}\left(V_{11}-V_{22}\right)B_{12}^W - \frac{P^\alpha}{M^\alpha}\frac{\partial B_{12}^W}{\partial R^\alpha} - \frac{1}{2}\left(F_{11}^\alpha + F_{22}^\alpha\right)\frac{\partial B_{12}^W}{\partial P^\alpha} \\
& -\frac{i\hbar}{2}(F_{11}^\alpha + F_{22}^\alpha)\frac{d_{12}^\alpha \Delta}{M^\alpha(V_{11}-V_{22})} \\
& -i\hbar\frac{P^\alpha}{M^\alpha}\frac{P^\gamma d_{12}^\gamma}{M^\gamma}\frac{\Delta}{(V_{11}-V_{22})^2}(F_{11}^\alpha - F_{22}^\alpha) \\
& -i\hbar\frac{P^\alpha}{M^\alpha}\frac{\partial d_{12}^\gamma}{\partial R^\alpha}\frac{P^\gamma}{M^\gamma}\frac{\Delta}{V_{11}-V_{22}} \\
& -\frac{i\hbar}{2}\frac{P^\alpha}{M^\alpha}\frac{\partial d_{12}^\gamma}{\partial R^\alpha}\frac{\partial \Gamma}{\partial P^\gamma} \\
& +i\hbar\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\left(\frac{4P^\alpha}{M^\alpha}Re\left(B_{12}^W d_{21}^\alpha\right)\right) \\
& +\frac{i\hbar}{2}\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\left((F_{22}^\alpha - F_{11}^\alpha)\frac{\partial \Delta}{\partial P^\alpha}\right) \\
& +\frac{i\hbar}{2}\frac{d_{12}^\gamma}{M^\gamma}\frac{\partial \Gamma}{\partial R^\gamma} \\
& +\frac{i\hbar}{4}d_{12}^\gamma\left((F_{22}^\alpha - F_{11}^\alpha)\frac{\partial^2 \Delta}{\partial P^\alpha \partial P^\gamma}\right) \\
& +i\hbar d_{12}^\gamma Re\left(\frac{\partial^2 B_{12}^W}{\partial P^\alpha \partial P^\gamma}F_{21}^\alpha\right) \\
& +i\hbar\frac{P^\gamma d_{12}^\gamma}{M^\gamma(V_{11}-V_{22})}\left[2\hbar Im\left(d_{21}^\alpha\frac{P^\gamma}{M^\gamma}d_{12}^\gamma\right)\frac{\partial \Gamma}{\partial P^\alpha}\right] \\
& -i\hbar d_{12}^\gamma\left[\hbar Im\left(\frac{d_{12}^\alpha d_{21}^\gamma}{M^\gamma}\right)\frac{\partial \Delta}{\partial P^\alpha}\right] \\
& -i\hbar d_{12}^\gamma\left[\hbar Im\left(d_{12}^\alpha\frac{P^\gamma}{M^\gamma}d_{21}^\gamma\right)\frac{\partial^2 \Delta}{\partial P^\alpha \partial P^\gamma}\right]
\end{aligned}
\tag{S24}
$$